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Hydromechanics of low-Reynolds-number flow. Part 3. Motion of a spheroidal particle in quadratic flows

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Exact solutions in closed form have been found using the singularity method for various quadratic flows of an unbounded incompressible viscous fluid at low Reynolds numbers past a prolate spheroid with an arbitrary orientation with respect to the fluid. The quadratic flows considered here include unidirectional paraboloidal flows, with either an elliptic or a hyperbolic velocity distribution, and stagnation-like quadratic flows as typical representations. The motion of a force-free spheroidal particle in a paraboloidal flow has been determined. It is shown that the spheroid rotates about three principal axes with angular velocities governed by a set of Jeffery orbital equations with the rate of shear evaluated at the centre of the spheroid. These angular velocities depend on the minor-to-major axis ratio of the spheroid and its instantaneous orientation, but are independent of its actual size. The spheroid also translates at a variable speed, depending on its orientation relative to the surrounding fluid, along a straight path parallel to the main flow direction without any side drift or migration. This 'jerk' motion obeys a trajectory equation which is size dependent.

1. Introduction

An exact solution for the motion of a spheroidal particle placed in a quadratic as well as in a linear flow of incompressible viscous fluid is very useful in the study of blood flow and general suspension rheology. In particular, a correct description of the behaviour of a spheroidal particle in a paraboloidal or Poiseuille flow will facilitate accurate calculation of the bulk flow properties of tube flows of dilute or concentrated suspensions of blood cells, long-chain polymers or any other biological supra-macromolecules. When the Reynolds number based on the particle size, the local flow velocity and the kinematic viscosity of the surrounding fluid is very small, as in the case of microcirculation of blood cells, the inertial effects of the fluid can be neglected and the Navier–Stokes equations of motion reduce to the Stokes equations as a first approximation.

Although the Stokes equations are linear, to obtain exact solutions to them for arbitrary body shapes or complicated flow conditions is still a formidable task. On the basis of a sophisticated analysis of ellipsoidal harmonics, Oberbeck (1876)

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solved the problem in which an ellipsoid translates through liquid at a constant speed in an arbitrary direction. Edwardes (1892), applying the same technique, obtained the solution for the steady motion of a viscous fluid in which an ellipsoid is constrained to rotate about a principal axis. The motion of an ellipsoidal particle in a general linear flow of viscous fluid at low Reynolds number has been solved by Jeffery (1922), whose solution was also built up using ellipsoidal harmonics. Adopting the singularity method, a different approach from the classical treatment of ellipsoidal harmonics, Chwang & Wu (1974, 1975) obtained a number of exact solutions for both exterior and interior Stokes-flow problems involving spheroids, spheres and circular cylinders. The fundamental singularities used in constructing these exact solutions are the Stokeslet and its derivatives, such as the rotlet, stresslet and potential doublet, for exterior flow problems (see also Hancock 1953; Batchelor 1970) and the Stokeson and its derivatives, called the roton and stresson, for interior flow problems (Chwang & Wu 1975).

Comparing the singularity method with the boundary-value method, we find that the former is simpler and more versatile than the latter provided that enough is known about the types of singularities required and their distribution range and densities for the specific body shape and given free-stream conditions. To this end, Chwang & Wu (1975) gave in their conclusions a set of empirical, yet useful rules regarding the types of singularities required for constructing exact solutions for given flow conditions. These rules provide guidelines for future developments in the singularity method at low Reynolds numbers. In the special case of slender bodies, Chwang & Wu (1974) obtained an asymptotic relationship between the nose curvature and the singularity strength near the end of its axial distribution. This asymptotic relationship was subsequently found by Wu & Chwang (1974) to be valid also for slender bodies in potential flows. The results established so far seem to suggest that the distribution range of the singularities depends only on the geometry of a given body, not on the flow conditions. Thus the distribution range determined for plane-symmetric bodies in potential flows (Wu & Chwang 1974) may be valid in Stokes flows as well. The densities or strengths of the required singularities are dictated by the boundary conditions to be satisfied on the surface of a given body.

It was the simplicity and elegance of the singularity method which stimulated the present author to use it to determine the motion of a spheroidal particle in a general quadratic flow field. Apart from the classical solution for a sphere in an axisymmetric paraboloidal flow which was obtained by Simha (1936; see also Chwang & Wu 1975, § 9) there were no solutions available for a general quadratic flow past a spheroid because of the overwhelming analytical difficulties. However this difficulty has now been overcome by an effective application of the singularity method, as we shall see in §§ 2–6 of this paper.

A solution is given in § 2 for an unbounded longitudinal paraboloidal flow $\mathbf{u} = (\beta y^2 + z^2)\,\mathbf{e}_x$ ($\mathbf{u}$ being the velocity vector and $\mathbf{e}_x$ a unit vector in the longitudinal or x direction) past a prolate spheroid whose surface is given by $x^2/a^2 + y^2/b^2 + z^2/b^2 = 1$, where a is the semi-major axis and b the semi-minor axis. This paraboloidal flow may be either elliptic or hyperbolic depending whether the constant β is positive or negative. When β vanishes, the paraboloidal flow degenerates into a two-dimensional parabolic flow. For arbitrary positive values of β, it represents Hagen–Poiseuille flow through a pipe of elliptic cross-section. If $\beta = 1$, it becomes a paraboloidal flow of revolution which corresponds to Hagen–Poiseuille flow in a circular tube. Hyperbolic paraboloidal flow ($\beta < 0$) may not exist physically, but it can certainly serve as a local component of a more complicated flow field. The corresponding solution for a transverse paraboloidal flow past a prolate spheroid is presented in §3. In §§4 and 5, longitudinal and transverse stagnation-like quadratic flows past a prolate spheroid are analysed, respectively.

The motion of a spheroidal particle placed arbitrarily in a paraboloidal flow is determined in §6. It is discovered from this exact solution that the spheroid rotates about its centre according to a set of Jeffery orbital equations [(56) and (57)]. Moreover the spheroid as a whole moves in a straight line parallel to the flow direction, without any side drift, at a variable speed which is governed by a trajectory equation [equation (53)]. Thus the present exact solution demonstrates definitely that there is no side drift or migration of rigid spheroidal particles in a paraboloidal flow if the flow is unbounded and the inertial effects of the fluid are completely neglected. Finally, this author feels that the new phenomenon of jerking motion revealed by (53) may have interesting consequences in future investigations of blood flow.

2. Longitudinal paraboloidal flow past a prolate spheroid

We consider first a paraboloidal flow with velocity profile

$$\mathbf{U} = (\beta y^2 + z^2)\,\mathbf{e}_x, \tag{1}$$

$\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ being the base vectors in the x, y and z directions respectively, past a prolate spheroid (see figure 1 a)

$$x^2/a^2 + r^2/b^2 = 1 \quad (r^2 = y^2+z^2,\ a \geqslant b), \tag{2a}$$

whose focal length $2c$ and eccentricity e are defined by the usual relationship

$$c = (a^2-b^2)^{\frac{1}{2}} = ea \quad (0 \leqslant e < 1). \tag{2b}$$

The paraboloidal flow given by (1) is parallel to the x or longitudinal axis of the spheroid (2). The constant β in (1) further classifies the flow to be elliptic paraboloidal if β is positive or hyperbolic paraboloidal if β is negative. As β vanishes, (1) degenerates into a two-dimensional parabolic flow. In this and all subsequent sections we shall assume that the inertia of the fluid is completely negligible and the incompressible viscous fluid satisfies the Stokes equations of motion

$$\nabla.\mathbf{u} = 0, \quad \nabla p = \mu\nabla^2\mathbf{u}, \tag{3a, b}$$

where $\mathbf{u}$ is the velocity vector, p the pressure and μ the (constant) viscosity coefficient of the fluid.

Before we apply the singularity method to construct an exact solution for this problem, a few words about the fundamental singularities of the Stokes equations

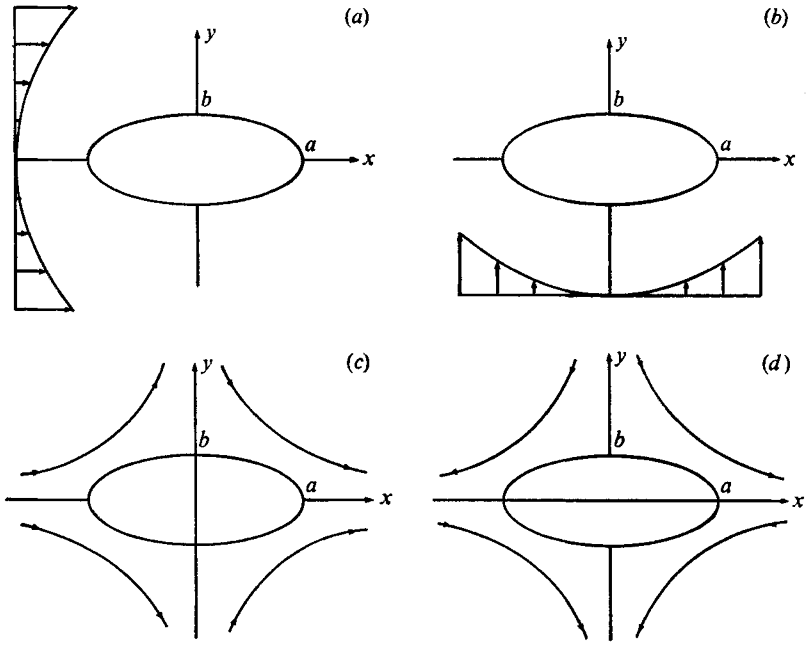

FIGURE 1. Schematic diagrams for (a) a longitudinal paraboloidal flow, (b) a transverse paraboloidal flow, (c) a longitudinal stagnation-like quadratic flow and (d) a transverse stagnation-like quadratic flow past a prolate spheroid.

may be helpful. The primary fundamental solution of (3), corresponding to a point force at the origin, is called a Stokeslet (Hancock 1953). The velocity $\mathbf{U}_S$, pressure P_S and vorticity $\boldsymbol{\zeta}_S = \nabla \times \mathbf{U}_S$ of a Stokeslet of strength $\boldsymbol{\alpha}$ are given by (see Batchelor 1970; Chwang & Wu 1975 for more details)

$$\mathbf{U}_S(\mathbf{x};\, \boldsymbol{\alpha}) = \boldsymbol{\alpha}/R + (\boldsymbol{\alpha}.\mathbf{x})\,\mathbf{x}/R^3 \quad (R = |\mathbf{x}|), \tag{4a}$$

$$P_S(\mathbf{x};\, \boldsymbol{\alpha}) = 2\mu\boldsymbol{\alpha}.\mathbf{x}/R^3, \tag{4b}$$

$$\boldsymbol{\zeta}_S(\mathbf{x};\, \boldsymbol{\alpha}) = 2\boldsymbol{\alpha} \times \mathbf{x}/R^3, \tag{4c}$$

where $\mathbf{x} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ is the position vector in three-dimensional Euclidean space. The total force exerted on this Stokeslet by the surrounding fluid is

$$\mathbf{F} = -8\pi\mu\boldsymbol{\alpha}. \tag{5}$$

Since the Stokes equations (3) are linear, a derivative of any order of a Stokeslet is also a solution of (3). In particular, the antisymmetric part of a Stokes doublet, called a rotlet, satisfies (3) by itself. A rotlet corresponds to a point torque at the origin. The velocity, pressure and vorticity of a rotlet of strength $\boldsymbol{\gamma}$ are given by

$$\mathbf{U}_R(\mathbf{x};\, \boldsymbol{\gamma}) = \tfrac{1}{2}\nabla \times \mathbf{U}_S(\mathbf{x};\, \boldsymbol{\gamma}) = \boldsymbol{\gamma} \times \mathbf{x}/R^3, \tag{6a}$$

$$P_R(\mathbf{x};\, \boldsymbol{\gamma}) = 0, \tag{6b}$$

$$\boldsymbol{\zeta}_R(\mathbf{x};\, \boldsymbol{\gamma}) = -\boldsymbol{\gamma}/R^3 + 3(\boldsymbol{\gamma}.\mathbf{x})\,\mathbf{x}/R^5 + 4\pi\boldsymbol{\gamma}\delta(\mathbf{x}), \tag{6c}$$

where $\delta(\mathbf{x})$ is the three-dimensional Dirac delta function. The total moment exerted on this rotlet by the surrounding fluid is

$$\mathbf{M} = -8\pi\mu\boldsymbol{\gamma}, \qquad 7)$$

while the total force on it is zero.

The symmetric part of a Stokes doublet produces another singularity, called a stresslet. Its velocity, pressure and vorticity are

$$\mathbf{U}_{SS}(\mathbf{x};\, \boldsymbol{\alpha}, \boldsymbol{\beta}) = [-\boldsymbol{\alpha}.\boldsymbol{\beta}/R^3 + 3(\boldsymbol{\alpha}.\mathbf{x})(\boldsymbol{\beta}.\mathbf{x})/R^5]\,\mathbf{x}, \tag{8a}$$

$$P_{SS}(\mathbf{x};\, \boldsymbol{\alpha}, \boldsymbol{\beta}) = 2\mu[-\boldsymbol{\alpha}.\boldsymbol{\beta}/R^3 + 3(\boldsymbol{\alpha}.\mathbf{x})(\boldsymbol{\beta}.\mathbf{x})/R^5], \tag{8b}$$

$$\boldsymbol{\zeta}_{SS}(\mathbf{x};\, \boldsymbol{\alpha}, \boldsymbol{\beta}) = 3R^{-5}[(\boldsymbol{\beta}.\mathbf{x})\,\boldsymbol{\alpha} + (\boldsymbol{\alpha}.\mathbf{x})\,\boldsymbol{\beta}] \times \mathbf{x}. \tag{8c}$$

In virtue of its symmetry, there is no net force or moment exerted on a stresslet by the surrounding fluid.

A potential doublet in Stokes flow can be obtained by contracting a Stokes quadrupole or simply by applying the Laplace operator to a Stokeslet as

$$\mathbf{U}_D(\mathbf{x};\boldsymbol{\delta}) = -\tfrac{1}{2}\nabla^2\mathbf{U}_S(\mathbf{x};\boldsymbol{\delta}) \quad (|\mathbf{x}| \neq 0). \tag{9a}$$

The velocity of a potential doublet given by (9*a*) may be rewritten as

$$\mathbf{U}_D(\mathbf{x};\, \boldsymbol{\delta}) = -\boldsymbol{\delta}/R^3 + 3(\boldsymbol{\delta}.\mathbf{x})\,\mathbf{x}/R^5, \tag{9b}$$

which is exactly the same as that of a doublet in potential flow. However, owing to the assumption of negligible inertia, a potential doublet in Stokes flow exerts no pressure, that is

$$P_D(\mathbf{x};\, \boldsymbol{\delta}) = 0. \tag{9c}$$

With the above singularities clarified, we now proceed to construct an exact solution for longitudinal paraboloidal flow past a prolate spheroid from a suitable distribution of these singularities and their derivatives. Since the average of the velocity (1) over the spheroidal surface (2) is

$$\frac{(1+\beta)\,a^2(1-e^2)\,[e(1+2e^2)\,(1-e^2)^{\frac{1}{2}} + (4e^2-1)\sin^{-1}e]\,\mathbf{e}_x}{8e^2[e(1-e^2)^{\frac{1}{2}} + \sin^{-1}e]}, \tag{10}$$

which does not vanish, a line distribution of Stokeslets in the x direction is needed. According to the rules of Chwang & Wu (1975), potential doublets are required whenever Stokeslets are present. As the y derivative of (1) gives a shear flow $2y\mathbf{e}_x$ and a shear-flow solution (see Chwang & Wu 1975, § 4) requires a line distribution of stresslets, rotlets and potential quadrupoles, we need here a line distribution of y derivatives of these singularities. Similarly, we need z derivatives of the singularities associated with the shear flow $2z\mathbf{e}_x$. All these singularities are, of course, distributed along the x axis between the foci $x = \pm c$. For the density of each individual singularity, we assume, from experience, a constant density for Stokeslets, a parabolic density for potential doublets, quartic densities for all Stokes quadrupoles and sextic densities for all potential octupoles. On this basis we find the solution to be of the form

$$\begin{aligned}\mathbf{u} = {} & (\beta y^2 + z^2)\,\mathbf{e}_x + A_1\int_{-c}^{c} \mathbf{U}_S(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x)\,d\xi + B_1\int_{-c}^{c}(c^2-\xi^2)\,\mathbf{U}_D(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x)\,d\xi \\ & + \int_{-c}^{c}(c^2-\xi^2)^2\left[C_1\frac{\partial}{\partial y}\mathbf{U}_{SS}(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x, \mathbf{e}_y) + D_1\frac{\partial}{\partial y}\mathbf{U}_R(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_z)\right. \\ & \left. + E_1\frac{\partial}{\partial z}\mathbf{U}_{SS}(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x, \mathbf{e}_z) + F_1\frac{\partial}{\partial z}\mathbf{U}_R(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_y)\right]d\xi \\ & + \int_{-c}^{c}(c^2-\xi^2)^3\left[G_1\frac{\partial^2}{\partial y^2} + H_1\frac{\partial^2}{\partial z^2}\right]\mathbf{U}_D(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x)\,d\xi,\end{aligned} \tag{11a}$$

$$p = 2(1+\beta)\mu x + A_1 \int_{-c}^{c} P_S(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x)\, d\xi$$
$$+\int_{-c}^{c} (c^2-\xi^2)^2 \left[C_1 \frac{\partial}{\partial y} P_{SS}(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x, \mathbf{e}_y) + E_1 \frac{\partial}{\partial z} P_{SS}(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x, \mathbf{e}_z)\right] d\xi, \quad (11b)$$

where (and below) $\boldsymbol{\xi} = \xi\mathbf{e}_x$, A_1, B_1, C_1, D_1, E_1, F_1, G_1 and H_1 are constants and the fundamental solutions $\mathbf{U}_S$, $\mathbf{U}_R$, $\mathbf{U}_{SS}$ and $\mathbf{U}_D$ are defined by (4), (6), (8) and (9) respectively. The first term $2(1+\beta)\mu x$ on the right-hand side of (11*b*) is the prevalent pressure corresponding to the given paraboloidal flow (1). Obviously the velocity and pressure given by (11) satisfy the Stokes equations (3) and also the boundary conditions on $\mathbf{u}$ and p at infinity. The boundary condition to be satisfied on the spheroidal surface (2*a*) is the no-slip condition

$$\mathbf{u} = 0 \quad \text{on} \quad x^2/a^2 + r^2/b^2 = 1. \quad (12)$$

Using (12), the constants A_1, B_1, etc., can be determined uniquely. Curtailing the details, which are straightforward although very cumbersome, we obtain the constants as

$$A_1 = -\frac{2e^2}{1-e^2} B_1 = \frac{(1+\beta)e^2a^2(1-e^2)}{3\{2e-(1+e^2)\log[(1+e)/(1-e)]\}}, \quad (13a)$$

$$C_1 = \frac{6e^2}{1-e^2} G_1, \quad E_1 = \frac{6e^2}{1-e^2} H_1, \quad (13b)$$

$$\begin{pmatrix} D_1 \\ F_1 \\ G_1 \\ H_1 \end{pmatrix} = \begin{pmatrix} -T_1 & T_1 & T_2 & T_2 \\ T_3 & T_3 & T_4 & -T_4 \\ 0 & T_5 & T_6 & T_7 \\ T_5 & 0 & T_7 & T_6 \end{pmatrix}^{-1} \begin{pmatrix} -1-\beta \\ 1-\beta \\ 0 \\ 0 \end{pmatrix}, \quad (13c)$$

$$T_1 = \frac{6}{1-e^2}\left[6e-(3-e^2)\log\frac{1+e}{1-e}\right], \quad (13d)$$

$$T_2 = \frac{36}{(1-e^2)^2}\left[2e(15-4e^2)-3(5-3e^2)\log\frac{1+e}{1-e}\right], \quad (13e)$$

$$T_3 = -\frac{2e(3-5e^2)}{(1-e^2)^2} + 3\log\frac{1+e}{1-e}, \quad (13f)$$

$$T_4 = T_6 - \frac{6e^2}{1-e^2} T_3, \quad (13g)$$

$$T_5 = 4\left[\frac{2e(3-2e^2)}{1-e^2} - 3\log\frac{1+e}{1-e}\right], \quad (13h)$$

$$T_6 = \frac{6}{1-e^2}\left[\frac{2e(15-13e^2)}{1-e^2} - 3(5-e^2)\log\frac{1+e}{1-e}\right], \quad (13i)$$

$$T_7 = 3(1-e^2)T_4 + \frac{12e^2}{1-e^2} T_5. \quad (13j)$$

The total force acting on the spheroid (2) can be obtained from (5) and (13*a*) as

$$\mathbf{F} = -8\pi\mu \int_{-c}^{c} A_1 \mathbf{e}_x\, d\xi = \frac{16\pi\mu(1+\beta)e^3a^3(1-e^2)\mathbf{e}_x}{3\{-2e+(1+e^2)\log[(1+e)/(1-e)]\}}. \quad (14)$$

This force, or drag, may be regarded as associated with that on the same prolate spheroid in a uniform flow with an equivalent velocity (see Chwang & Wu 1975, equation (28) for the drag on a prolate spheroid in a uniform flow)

$$\mathbf{U}_e = \tfrac{1}{3}(1+\beta)a^2(1-e^2)\mathbf{e}_x, \tag{15}$$

which is, however, different from the surface-average velocity (10). Only when the spheroid (2) degenerates into a sphere, that is $e \to 0$ (or $a \to b$), does the equivalent velocity (15) become the same as the surface-average velocity (10), both having the value $\frac{1}{3}(1+\beta)a^2$.

In the limiting case of a sphere ($e \to 0$), we deduce from the above solution that as $e \to 0$

$$\int_{-c}^{c} A_1\,d\xi = -\frac{(1+\beta)a^3}{4}, \qquad \int_{-c}^{c} B_1(c^2-\xi^2)\,d\xi = \frac{(1+\beta)a^5}{12},$$

$$\int_{-c}^{c} C_1(c^2-\xi^2)^2 d\xi = \frac{7\beta a^5}{24}, \qquad \int_{-c}^{c} D_1(c^2-\xi^2)^2 d\xi = \frac{(2-\beta)a^5}{24},$$

$$\int_{-c}^{c} E_1(c^2-\xi^2)^2 d\xi = \frac{7a^5}{24}, \qquad \int_{-c}^{c} F_1(c^2-\xi^2)^2 d\xi = \frac{(1-2\beta)a^5}{24},$$

$$\int_{-c}^{c} G_1(c^2-\xi^2)^3 d\xi = \frac{\beta a^7}{24}, \qquad \int_{-c}^{c} H_1(c^2-\xi^2)^3 d\xi = \frac{a^7}{24}.$$

Consequently, applying the identity

$$\frac{\partial}{\partial z}\mathbf{U}_R(\mathbf{x};\,\mathbf{e}_y) - \frac{\partial}{\partial y}\mathbf{U}_R(\mathbf{x};\,\mathbf{e}_z) = \mathbf{U}_D(\mathbf{x};\,\mathbf{e}_x), \tag{16}$$

we have the velocity $\mathbf{u}$ and drag $\mathbf{F}$ on a sphere of radius a in a paraboloidal flow (1) as

$$\mathbf{u} = (\beta y^2+z^2)\,\mathbf{e}_x - \frac{(1+\beta)a^3}{4}\left(\frac{\mathbf{e}_x}{R}+\frac{x\mathbf{x}}{R^3}\right) + \frac{7\beta a^5}{24}\frac{\partial}{\partial y}\left(\frac{3xy\mathbf{x}}{R^5}\right) - \frac{\beta a^5}{8}\frac{\partial}{\partial y}\left(\frac{\mathbf{e}_z\times\mathbf{x}}{R^3}\right)$$
$$+\frac{7a^5}{24}\frac{\partial}{\partial z}\left(\frac{3xz\mathbf{x}}{R^5}\right) + \frac{a^5}{8}\frac{\partial}{\partial z}\left(\frac{\mathbf{e}_y\times\mathbf{x}}{R^3}\right) + \frac{a^7}{24}\left(\beta\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(-\frac{\mathbf{e}_x}{R^3}+\frac{3x\mathbf{x}}{R^5}\right), \tag{17a}$$

$$\mathbf{F} = 2\pi\mu(1+\beta)a^3\mathbf{e}_x. \tag{17b}$$

For an axisymmetric paraboloidal flow ($\beta = 1$), (17) further reduce to

$$\mathbf{u} = r^2\mathbf{e}_x - \frac{a^3}{2}\left(1-\frac{7a^2}{12}\frac{\partial^2}{\partial x^2}\right)\left(\frac{\mathbf{e}_x}{R}+\frac{x\mathbf{x}}{R^3}\right) + \frac{a^5}{12}\left(5-\frac{a^2}{2}\frac{\partial^2}{\partial x^2}\right)\nabla\frac{\partial}{\partial x}\left(\frac{1}{R}\right), \tag{18a}$$

$$\mathbf{F} = 4\pi\mu a^3\mathbf{e}_x, \tag{18b}$$

which agree with the result of Simha (1936) and also with equations (68) of Chwang & Wu (1975). In obtaining (18) from (17), we have applied the identities

$$\frac{\partial}{\partial y}\mathbf{U}_{SS}(\mathbf{x};\,\mathbf{e}_x,\mathbf{e}_y) + \frac{\partial}{\partial z}\mathbf{U}_{SS}(\mathbf{x};\,\mathbf{e}_x,\mathbf{e}_z) = \frac{\partial^2}{\partial x^2}\mathbf{U}_S(\mathbf{x};\,\mathbf{e}_x) + \mathbf{U}_D(\mathbf{x};\,\mathbf{e}_x), \tag{19}$$

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\mathbf{U}_D(\mathbf{x};\,\mathbf{e}_x) = -\frac{\partial^2}{\partial x^2}\mathbf{U}_D(\mathbf{x};\,\mathbf{e}_x) \quad (|\mathbf{x}| \neq 0) \tag{20}$$

in addition to identity (16).

3. Transverse paraboloidal flow past a prolate spheroid

We next consider a transverse paraboloidal flow with velocity profile

$$\mathbf{U} = (\alpha x^2 + z^2)\,\mathbf{e}_y \tag{21}$$

past the prolate spheroid specified by (2) (see figure 1*b*). The constant α in (21) is arbitrary. As in the solution in § 2, we need here a line distribution of Stokeslets, potential doublets, Stokes quadrupoles (derivatives of stresslets and rotlets) and potential octupoles. The exact solution for the velocity $\mathbf{u}$ and pressure p can be expressed as

$$\begin{aligned}\mathbf{u} = {} & (\alpha x^2 + z^2)\,\mathbf{e}_y + A_2 \int_{-c}^{c} \mathbf{U}_S(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_y)\, d\xi + B_2 \int_{-c}^{c} (c^2-\xi^2)\, \mathbf{U}_D(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_y)\, d\xi \\ & + \int_{-c}^{c} (c^2-\xi^2)^2 \Big[C_2 \frac{\partial}{\partial x} \mathbf{U}_{SS}(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x, \mathbf{e}_y) + D_2 \frac{\partial}{\partial x} \mathbf{U}_R(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_z) \\ & + E_2 \frac{\partial}{\partial z} \mathbf{U}_{SS}(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_y, \mathbf{e}_z) + F_2 \frac{\partial}{\partial z} \mathbf{U}_R(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x) \Big] d\xi \\ & + \int_{-c}^{c} (c^2-\xi^2)^3 \left[G_2 \frac{\partial^2}{\partial x^2} + H_2 \frac{\partial^2}{\partial z^2} \right] \mathbf{U}_D(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_y)\, d\xi, \end{aligned} \tag{22a}$$

$$\begin{aligned} p = {} & 2(1+\alpha)\mu y + A_2 \int_{-c}^{c} P_S(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_y)\, d\xi \\ & + \int_{-c}^{c} (c^2-\xi^2)^2 \left[C_2 \frac{\partial}{\partial x} P_{SS}(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x, \mathbf{e}_y) + E_2 \frac{\partial}{\partial z} P_{SS}(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_y, \mathbf{e}_z) \right] d\xi. \end{aligned} \tag{22b}$$

The eight new constants $A_2, \ldots, H_2$ can be determined from the no-slip condition (12), which gives

$$A_2 = -\frac{2e^2}{1-e^2} B_2 = -\frac{2e^2 a^2 (1+\alpha-e^2)}{3\{2e + (3e^2-1)\log[(1+e)/(1-e)]\}}, \tag{23a}$$

$$C_2 = \frac{6e^2}{1-e^2} G_2, \quad E_2 = \frac{6e^2}{1-e^2} H_2, \tag{23b}$$

$$F_2 = T_6(G_2 - \tfrac{1}{2} H_2)/T_3, \quad D_2 = -(Q_1 G_2 + T_6 H_2)/T_5, \tag{23c}$$

$$G_2 = \left[\left(Q_4 - \frac{T_6 Q_2}{T_5} - \frac{T_6 Q_3}{2T_3} \right) H_2 - \alpha \right] \Big/ \left(42 Q_3 - \frac{T_6 Q_3}{T_3} + \frac{Q_1 Q_2}{T_5} \right), \tag{23d}$$

$$H_2 = \frac{1-e^2}{6} \left[\frac{e(15 - 34e^2 + 23e^4)}{(1-e^2)^2} - \frac{3(5-3e^2)}{2} \log\frac{1+e}{1-e} \right]^{-1}, \tag{23e}$$

where the T's are given by (13) and the Q's by

$$Q_1 = -\frac{18e^2}{1-e^2} T_5 - 4(1-e^2) T_4, \quad Q_2 = -(1-e^2) T_1, \quad Q_3 = -\frac{1-e^2}{12} T_6, \tag{24a}$$

$$Q_4 = 3\left[-\frac{e(45 - 78e^2 + 29e^4)}{(1-e^2)^2} + \frac{3(15-e^2)}{2} \log\frac{1+e}{1-e} \right]. \tag{24b}$$

The total drag on the prolate spheroid may be obtained easily from (5) and (23*a*), which give

$$\mathbf{F} = -8\pi\mu \int_{-c}^{c} A_2 \mathbf{e}_y\, d\xi = \frac{32\pi\mu e^3 a^3 (1+\alpha-e^2)\, \mathbf{e}_y}{3\{2e + (3e^2-1)\log[(1+e)/(1-e)]\}}. \tag{25}$$

This drag could be regarded as that experienced by the prolate spheroid (2) when placed in a uniform flow of equivalent velocity

$$\mathbf{U}_e = \tfrac{1}{3}a^2(1+\alpha-e^2)\,\mathbf{e}_y. \tag{26}$$

In the limiting case of a sphere ($e \to 0$), (25) reduces to

$$\lim_{e\to 0} \mathbf{F} = 2\pi\mu(1+\alpha)\,a^3\mathbf{e}_y. \tag{27 a}$$

In the meantime, the velocity $\mathbf{u}$, given by (22 *a*), becomes

$$\begin{aligned}\mathbf{u} = (\alpha x^2+z^2)\,\mathbf{e}_y - \frac{(1+\alpha)\,a^3}{4}\,\mathbf{U}_S(\mathbf{x};\,\mathbf{e}_y) + \frac{7\alpha a^5}{24}\frac{\partial}{\partial x}\mathbf{U}_{SS}(\mathbf{x};\,\mathbf{e}_x,\mathbf{e}_y)\\ +\frac{\alpha a^5}{8}\frac{\partial}{\partial x}\mathbf{U}_R(\mathbf{x};\,\mathbf{e}_z) + \frac{7a^5}{24}\frac{\partial}{\partial z}\mathbf{U}_{SS}(\mathbf{x};\,\mathbf{e}_y,\mathbf{e}_z) - \frac{a^5}{8}\frac{\partial}{\partial z}\mathbf{U}_R\,(\mathbf{x};\,\mathbf{e}_x)\\ +\frac{a^7}{24}\left(\alpha\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\mathbf{U}_D(\mathbf{x};\,\mathbf{e}_y).\end{aligned} \tag{27 b}$$

In obtaining the above expression for the velocity field around a sphere, we have used the identity

$$\frac{\partial}{\partial x}\mathbf{U}_R(\mathbf{x};\,\mathbf{e}_z) - \frac{\partial}{\partial z}\mathbf{U}_R(\mathbf{x};\,\mathbf{e}_x) = \mathbf{U}_D(\mathbf{x};\,\mathbf{e}_y). \tag{28}$$

As a matter of fact, (27 *b*) is a counterpart of expression (17 *a*).

4. Longitudinal stagnation-like quadratic flow past a spheroid

We now consider the problem of a spheroid given by (2) placed in a longitudinal stagnation-like quadratic flow with velocity (see figure 1 *c*)

$$\mathbf{U} = x^2\mathbf{e}_x - 2xy\mathbf{e}_y, \tag{29}$$

which obviously satisfies the continuity equation (3 *a*) and the momentum equation (3 *b*) if the pressure associated with it is $2\mu x$. The stagnation plane is the centre-plane $x = 0$. In the half-space $x < 0$ the flow is towards the stagnation plane while in the half-space $x > 0$ it is away from this plane. This type of quadratic flow is important since it can serve as a component in the general study of the motion of a spheroidal particle placed in a paraboloidal flow whose direction does not coincide with any one of the principal axes of the spheroid (see §6 of this paper). The exact solution for the velocity field involves a line distribution, between the foci $x = \pm c$, of Stokeslets, potential doublets and some Stokes quadrupoles and potential octupoles with constant, parabolic, quartic and sextic densities respectively. Thus the velocity $\mathbf{u}$ and pressure p take the following forms:

$$\begin{aligned}\mathbf{u} = x^2\mathbf{e}_x - 2xy\mathbf{e}_y + A_3\int_{-c}^{c}\mathbf{U}_S(\mathbf{x}-\boldsymbol{\xi};\,\mathbf{e}_x)\,d\xi + B_3\int_{-c}^{c}(c^2-\xi^2)\,\mathbf{U}_D(\mathbf{x}-\boldsymbol{\xi};\,\mathbf{e}_x)\,d\xi\\ +\int_{-c}^{c}(c^2-\xi^2)^2\left[C_3\frac{\partial}{\partial x}\mathbf{U}_{SS}(\mathbf{x}-\boldsymbol{\xi};\,\mathbf{e}_x,\mathbf{e}_x) + D_3\frac{\partial}{\partial x}\mathbf{U}_{SS}(\mathbf{x}-\boldsymbol{\xi};\,\mathbf{e}_y,\mathbf{e}_y)\right.\\ \left. + G_3\frac{\partial}{\partial y}\mathbf{U}_{SS}(\mathbf{x}-\boldsymbol{\xi};\,\mathbf{e}_x,\mathbf{e}_y)\right]d\xi\\ +\int_{-c}^{c}(c^2-\xi^2)^3\left[E_3\frac{\partial^2}{\partial x^2}\mathbf{U}_D(\mathbf{x}-\boldsymbol{\xi};\,\mathbf{e}_x) + F_3\frac{\partial^2}{\partial x\,\partial y}U_D(\mathbf{x}-\boldsymbol{\xi};\,\mathbf{e}_y)\right]d\xi\end{aligned} \tag{30 a}$$

and

$$p = 2\mu x + A_3 \int_{-c}^{c} P_S(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x)\, d\xi + \int_{-c}^{c} (c^2-\xi^2)^2 \left[C_3 \frac{\partial}{\partial x} P_{SS}(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x, \mathbf{e}_x) \right.$$

$$\left. + D_3 \frac{\partial}{\partial x} P_{SS}(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_y, \mathbf{e}_y) + G_3 \frac{\partial}{\partial y} P_{SS}(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x, \mathbf{e}_y) \right] d\xi. \quad (30\,b)$$

The seven constants $A_3, \ldots, G_3$ are determined uniquely by the no-slip boundary condition (12) as

$$A_3 = \frac{e^2a^2}{3\{2e-(1+e^2)\log[(1+e)/(1-e)]\}}, \quad D_3 = -\frac{T_4 F_3}{T_3}, \quad (31\,a)$$

$$G_3 = \frac{T_6 F_3}{T_3}, \quad B_3 = -\frac{1-e^2}{2e^2} A_3 + 4(1-e^2)\, a^2 C_3 - 24e^2a^2E_3, \quad (31\,b)$$

$$E_3 = (N_4 C_3 + N_6 F_3)/N_5, \quad F_3 = [(1-e^2)\,T_4 + T_5 T_6/2T_3]^{-1}, \quad (31\,c)$$

$$C_3 = \frac{(\frac{3}{2}T_6 N_5 - N_3 N_6)\, F_3 - (1+N_2 D_3)\, N_5}{N_3 N_4 + N_1 N_5}, \quad (31\,d)$$

where the T's are defined in (13) and the (constant) N's by

$$N_1 = 24\left[4e-(2-e^2)\log\frac{1+e}{1-e}\right], \quad N_2 = 3\left[\frac{e(3-e^2)}{1-e^2} - \frac{3+e^2}{2}\log\frac{1+e}{1-e}\right], \quad (32\,a)$$

$$N_3 = 30(1-e^2)\,T_1, \quad N_4 = 16e^3/(1-e^2) - 2T_5, \quad (32\,b)$$

$$N_5 = \frac{96e^5}{(1-e^2)^2} + 4(1-e^2)\,T_4, \quad N_6 = \frac{6e^2}{1-e^2}T_5 + (1-e^2)\,T_4. \quad (32\,c)$$

The total drag force on the spheroid is evaluated from (5) and (31 a), which give

$$\mathbf{F} = -8\pi\mu \int_{-c}^{c} A_3 \mathbf{e}_x\, d\xi = \frac{16\pi\mu e^3 a^3 \mathbf{e}_x}{3\{-2e+(1+e^2)\log[(1+e)/(1-e)]\}}, \quad (33)$$

which reduces in the limiting case of a sphere ($e \to 0$) to the following:

$$\lim_{e\to 0} \mathbf{F} = 2\pi\mu a^3 \mathbf{e}_x. \quad (34)$$

The corresponding velocity field for a sphere can be deduced from (30 a) as

$$\mathbf{u} = x^2\mathbf{e}_x - 2xy\mathbf{e}_y - \frac{a^3}{4}\left(\frac{\mathbf{e}_x}{R} + \frac{x\mathbf{x}}{R^3}\right) + \frac{7a^5}{24}\frac{\partial}{\partial x}\left(-\frac{\mathbf{x}}{R^3} + \frac{3x^2\mathbf{x}}{R^5}\right) - \frac{5a^5}{12}\frac{\partial}{\partial x}\left(-\frac{\mathbf{x}}{R^3} + \frac{3y^2\mathbf{x}}{R^5}\right)$$

$$-\frac{a^5}{6}\frac{\partial}{\partial y}\left(\frac{3xy\mathbf{x}}{R^5}\right) + \frac{a^7}{24}\frac{\partial^2}{\partial x^2}\left(-\frac{\mathbf{e}_x}{R^3} + \frac{3x\mathbf{x}}{R^5}\right) - \frac{a^7}{12}\frac{\partial^2}{\partial x\,\partial y}\left(-\frac{\mathbf{e}_y}{R^3} + \frac{3y\mathbf{x}}{R^5}\right). \quad (35)$$

5. Transverse stagnation-like quadratic flow past a spheroid

For a transverse stagnation-like quadratic flow with velocity profile

$$\mathbf{U} = 2xy\mathbf{e}_x - y^2\mathbf{e}_y \quad (36)$$

past a spheroid (see figure 1 d), the solution is analogous to that of the previous section. Since the y derivative $2x\mathbf{e}_x - 2y\mathbf{e}_y$ of (36) represents a two-dimensional extensional flow and a two-dimensional extensional flow needs stresslets and potential quadrupoles for constructing an exact solution, we require the y derivatives of these singularities for the present problem. As the main flow (36) has a net streaming motion in the $-y$ direction, Stokeslets and their associated potential doublets, both in the y direction, are required. All the above singularities are distributed along the x axis between the foci $x = \pm c$ of the given spheroid (2). Thus the solution can be expressed as

$$\begin{aligned}\mathbf{u} = 2xy\mathbf{e}_x - y^2\mathbf{e}_y + A_4\int_{-c}^{c} \mathbf{U}_S(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_y)\,d\xi + B_4\int_{-c}^{c}(c^2-\xi^2)\,\mathbf{U}_D(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_y)\,d\xi \\ + \int_{-c}^{c}(c^2-\xi^2)^2\left[C_4\frac{\partial}{\partial y}\mathbf{U}_{SS}(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x, \mathbf{e}_x) + D_4\frac{\partial}{\partial y}\mathbf{U}_{SS}(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_y, \mathbf{e}_y)\right. \\ \left. + G_4\frac{\partial}{\partial x}\mathbf{U}_{SS}(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x, \mathbf{e}_y)\right]d\xi \\ + \int_{-c}^{c}(c^2-\xi^2)^3\left[E_4\frac{\partial^2}{\partial x\,\partial y}\mathbf{U}_D(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x) + F_4\frac{\partial^2}{\partial y^2}\mathbf{U}_D(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_y)\right]d\xi,\end{aligned} \qquad (37\,a)$$

$$\begin{aligned}p = -2\mu y + A_4\int_{-c}^{c} P_S(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_y)\,d\xi + \int_{-c}^{c}(c^2-\xi^2)^2\left[C_4\frac{\partial}{\partial y}P_{SS}(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x, \mathbf{e}_x)\right. \\ \left. + D_4\frac{\partial}{\partial y}P_{SS}(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_y, \mathbf{e}_y) + G_4\frac{\partial}{\partial x}P_{SS}(\mathbf{x}-\boldsymbol{\xi};\, \mathbf{e}_x, \mathbf{e}_y)\right]d\xi.\end{aligned} \qquad (37\,b)$$

By applying the no-slip boundary condition (12) on the spheroidal surface (2), we determine the seven unknown constants $A_4, \ldots, G_4$ in (37) as

$$A_4 = \frac{2e^2a^2(1-e^2)}{3\{2e+(3e^2-1)\log[(1+e)/(1-e)]\}}, \qquad (38\,a)$$

$$D_4 = \frac{6e^2}{1-e^2}F_4 = 2e^2\left[-\frac{2e(15-34e^2+23e^4)}{(1-e^2)^2} + 3(5-3e^2)\log\frac{1+e}{1-e}\right]^{-1}, \qquad (38\,b)$$

$$B_4 = 4a^2(1-e^2)(C_4+G_4) - 24e^2a^2E_4 - \frac{1-e^2}{2e^2}A_4, \qquad (38\,c)$$

$$G_4 = -C_4 + (1+T_6F_4 + 30T_3E_4)/K_2, \qquad (38\,d)$$

$$C_4 = (K_1F_4 - \tfrac{5}{2}T_6E_4)/T_1, \qquad (38\,e)$$

$$E_4 = (2 + K_4/K_2 + K_6F_4)/K_5, \qquad (38\,f)$$

where the (constant) K's are given by

$$K_1 = \frac{3}{(1-e^2)^2}\left[5e(21-11e^2) - \tfrac{3}{2}(35-30e^2+3e^4)\log\frac{1+e}{1-e}\right], \qquad (39\,a)$$

$$K_2 = 12\left[\frac{2e}{1-e^2} - \log\frac{1+e}{1-e}\right], \quad K_3 = \frac{16e^3}{1-e^2} - 4T_5, \quad K_4 = K_3 + T_5, \qquad (39\,b)$$

$$K_5 = N_5 - \frac{5T_5T_6}{2T_1} - \frac{30T_3K_4}{K_2}, \qquad (39\,c)$$

$$K_6 = \frac{18e^2}{1-e^2}T_5 + \frac{K_4T_6}{K_2} - \frac{T_5K_1}{T_1} + 3(1-e^2)T_4, \qquad (39\,d)$$

with N_5 and $T_1, \ldots, T_6$ defined in (32 *c*) and (13 *d*–*i*) respectively.

The total drag force exerted on the given spheroid by the surrounding fluid can be obtained from (5) and (38 *a*):

$$\mathbf{F} = -8\pi\mu\int_{-c}^{c} A_4\mathbf{e}_y\,d\xi = -\frac{32\pi\mu e^3a^3(1-e^2)\,\mathbf{e}_y}{3\{2e+(3e^2-1)\log[(1+e)/(1-e)]\}}. \tag{40}$$

This drag is in the $-y$ direction, as it should be since the main stagnation-like quadratic flow has a net streaming motion in the $-y$ direction.

In the limiting case of a sphere ($e\to 0$), the velocity $\mathbf{u}$ and drag $\mathbf{F}$ reduce to

$$\begin{aligned}\mathbf{u} = {} & 2xy\mathbf{e}_x - y^2\mathbf{e}_y + \frac{a^3}{4}\left(\frac{\mathbf{e}_y}{R}+\frac{y\mathbf{x}}{R^3}\right)+\frac{5a^5}{12}\frac{\partial}{\partial y}\left(-\frac{\mathbf{x}}{R^3}+\frac{3x^2\mathbf{x}}{R^5}\right)\\ & -\frac{7a^5}{24}\frac{\partial}{\partial y}\left(-\frac{\mathbf{x}}{R^3}+\frac{3y^2\mathbf{x}}{R^5}\right)+\frac{a^5}{6}\frac{\partial}{\partial x}\left(\frac{3xy\mathbf{x}}{R^5}\right)+\frac{a^7}{12}\frac{\partial^2}{\partial x\,\partial y}\left(-\frac{\mathbf{e}_x}{R^3}+\frac{3x\mathbf{x}}{R^5}\right)\\ & -\frac{a^7}{24}\frac{\partial^2}{\partial y^2}\left(-\frac{\mathbf{e}_y}{R^3}+\frac{3y\mathbf{x}}{R^5}\right),\end{aligned} \tag{41}$$

$$\mathbf{F} = -2\pi\mu a^3\mathbf{e}_y. \tag{42}$$

It is noted that (41) and (42) become identical to (35) and (34) on replacing x, y, $\mathbf{e}_x$ and $\mathbf{e}_y$ in (34) and (35) by $-y$, x, $-\mathbf{e}_y$ and $\mathbf{e}_x$ respectively.

This solution corresponding to transverse stagnation-like quadratic flow past a prolate spheroid, like the one in § 4, plays an important role in determining the general motion of a spheroidal particle placed arbitrarily in a paraboloidal flow as discussed in the next section.

6. Motion of a spheroidal particle in a paraboloidal flow

With all the background knowledge about quadratic flows past a prolate spheroid provided by §§ 2–5, we now proceed to consider the motion of a prolate spheroid placed arbitrarily in a paraboloidal flow of unbounded incompressible viscous fluid. Let the surface of the spheroid be given again by

$$x^2/a^2+r^2/b^2 = 1 \quad (r^2 = y^2+z^2,\ a \geqslant b)$$

in the co-ordinate system (x,y,z) fixed with respect to the body (see figure 2, where the z axis is pointing out of the paper, towards the reader). Without loss of generality, let the paraboloidal flow be of the form

$$\mathbf{U} = K(\eta^2+\zeta^2)\,\mathbf{e}_\xi \tag{43 a}$$

in the co-ordinates (ξ,η,ζ), which are related to the co-ordinates (x,y,z) by

$$\xi = x\cos\theta+y\sin\theta, \quad \eta+h = -x\sin\theta+y\cos\theta, \quad \zeta = z, \tag{44}$$

where θ is the angle between the x and ξ axes and h is the distance between the centre O of the spheroid and the ξ axis.

The paraboloidal flow (43 *a*) may be expressed in the body co-ordinates (x, y, z) by using the relationship (44):

$$\begin{aligned}\mathbf{U} = {} & Kh^2(\cos\theta\mathbf{e}_x+\sin\theta\mathbf{e}_y)-2Kh(y\cos^2\theta\mathbf{e}_x-x\sin^2\theta\mathbf{e}_y)\\ & +2Kh\sin\theta\cos\theta\,(x\mathbf{e}_x-y\mathbf{e}_y)+K\cos\theta(y^2\cos^2\theta+z^2)\,\mathbf{e}_x\\ & +K\sin\theta(x^2\sin^2\theta+z^2)\,\mathbf{e}_y+K\sin^2\theta\cos\theta(x^2\mathbf{e}_x-2xy\mathbf{e}_y)\\ & -K\sin\theta\cos^2\theta(2xy\mathbf{e}_x-y^2\mathbf{e}_y).\end{aligned} \tag{43 b}$$

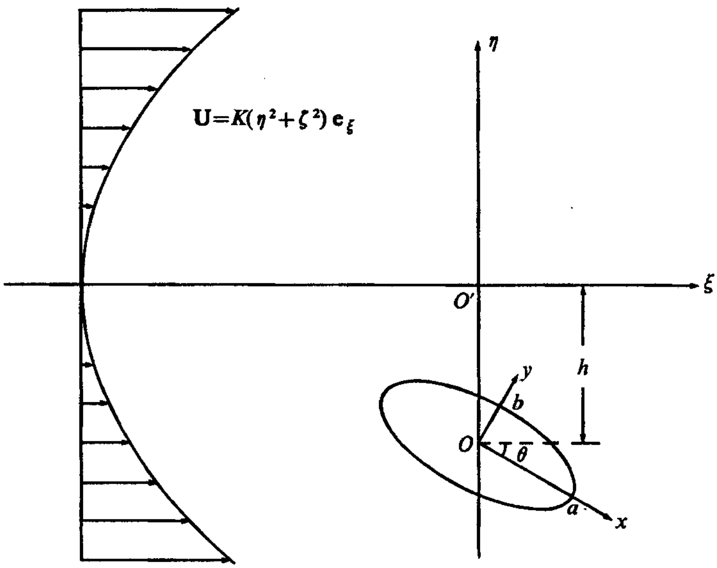

FIGURE 2. Relative orientation of a prolate spheroidal particle with respect to a paraboloidal flow of incompressible viscous fluid.

The first term on the right-hand side of (43*b*) represents a uniform flow; the drag force acting on a prolate spheroid (2) in this flow is (see Chwang & Wu 1975, § 3)

$$\mathbf{F}_1 = \frac{16\pi\mu e^3 aKh^2\cos\theta}{-2e+(1+e^2)\log[(1+e)/(1-e)]}\,\mathbf{e}_x + \frac{32\pi\mu e^3 aKh^2\sin\theta}{2e+(3e^2-1)\log[(1+e)/(1-e)]}\,\mathbf{e}_y. \tag{45}$$

The second term $-2Kh(y\cos^2\theta\mathbf{e}_x - x\sin^2\theta\mathbf{e}_y)$ on the right-hand side of (43*b*) designates shear flow. There is no force involved with this type of shear flow. However, in this shear flow the fluid exerts a moment or torque on the spheroid about the z axis. This moment can be evaluated easily to yield (see Chwang & Wu 1975, §§ 4 and 5)

$$\mathbf{M} = \frac{64\pi\mu e^3 a^3 Kh(1-e^2\cos^2\theta)}{3\{-2e+(1+e^2)\log[(1+e)/(1-e)]\}}\,\mathbf{e}_z. \tag{46}$$

The third term, $2Kh\sin\theta\cos\theta(x\mathbf{e}_x - y\mathbf{e}_y)$, characterizes a two-dimensional extensional flow. The exact solution for this type of flow past a prolate spheroid can be constructed without any difficulty in a manner similar to that discussed in § 8 of Chwang & Wu's (1975) paper for a three-dimensional extensional flow. Curtailing the details, we note that there is no net force or moment exerted on the spheroid by this type of flow owing to the symmetry of the body and of the flow.

Unlike the uniform or linear flow fields, the last four terms on the right-hand side of (43*b*) are all quadratic flows. The drag force associated with the longitudinal paraboloidal flow $K\cos\theta(y^2\cos^2\theta + z^2)\,\mathbf{e}_x$ is [see (14) with β replaced by $\cos^2\theta$]

$$\mathbf{F}_2 = \frac{16\pi\mu e^3 a^3(1-e^2)K\cos\theta(1+\cos^2\theta)}{3\{-2e+(1+e^2)\log[(1+e)/(1-e)]\}}\,\mathbf{e}_x. \tag{47}$$

From (25), the drag associated with the transverse paraboloidal flow

$$K \sin\theta (x^2 \sin^2\theta + z^2)\, \mathbf{e}_y$$

is found to be (replacing α by $\sin^2\theta$)

$$\mathbf{F}_3 = \frac{32\pi\mu e^3 a^3 K \sin\theta (1+\sin^2\theta - e^2)}{3\{2e + (3e^2-1)\log[(1+e)/(1-e)]\}} \mathbf{e}_y. \tag{48}$$

The contributions to the total drag from the last two terms of (43*b*), namely the longitudinal stagnation-like quadratic flow $K\sin^2\theta\cos\theta(x^2\mathbf{e}_x - 2xy\mathbf{e}_y)$ and the transverse stagnation-like quadratic flow $-K\sin\theta\cos^2\theta(2xy\mathbf{e}_x - y^2\mathbf{e}_y)$, are [see (33) and (40)]

$$\mathbf{F}_4 = \frac{16\pi\mu e^3 a^3 K \sin^2\theta\cos\theta}{3\{-2e + (1+e^2)\log[(1+e)/(1-e)]\}} \mathbf{e}_x \tag{49}$$

and

$$\mathbf{F}_5 = \frac{32\pi\mu e^3 a^3 (1-e^2) K \sin\theta\cos^2\theta}{3\{2e + (3e^2-1)\log[(1+e)/(1-e)]\}} \mathbf{e}_y \tag{50}$$

respectively. Combining (45) and (47)–(50), the total drag force on the spheroid is therefore

$$\mathbf{F} = \sum_{i=1}^{5} \mathbf{F}_i = \frac{16\pi\mu e^3 a K \cos\theta[h^2 + \frac{1}{3}a^2(2-e^2-e^2\cos^2\theta)]}{-2e + (1+e^2)\log[(1+e)/(1-e)]} \mathbf{e}_x + \frac{32\pi\mu e^3 a K \sin\theta[h^2 + \frac{1}{3}a^2(2-e^2-e^2\cos^2\theta)]}{2e + (3e^2-1)\log[(1+e)/(1-e)]} \mathbf{e}_y. \tag{51}$$

The total moment is given by (46).

If the spheroidal particle is allowed to move freely in the fluid, the resultant force and the resultant moment on the particle must both vanish at every instant. Since the Stokes equations are linear, these conditions can be satisfied only if the spheroidal particle rotates about the z axis with an angular velocity

$$\boldsymbol{\omega} = \frac{2Kh(1-e^2\cos^2\theta)}{(2-e^2)} \mathbf{e}_z \tag{52}$$

and in the meantime translates with a linear velocity

$$\mathbf{u} = [Kh^2 + \tfrac{1}{3}Ka^2(2-e^2-e^2\cos^2\theta)]\,\mathbf{e}_\xi, \tag{53}$$

where $\mathbf{e}_\xi$ is related to $\mathbf{e}_x$ and $\mathbf{e}_y$ by $\mathbf{e}_\xi = \cos\theta\mathbf{e}_x + \sin\theta\mathbf{e}_y$. The moment due to the rotation (52) of a prolate spheroid about its minor axis is of exactly the same magnitude as that in (46) but in the opposite direction (see Chwang & Wu 1975, §7 for rotation about a minor axis). Similarly, the drag due to translation (53) is of the same magnitude as and opposite to (51).

Equation (52) is the Jeffery (1922) orbital equation for the motion of a prolate spheroid in a shear flow with the shear rate evaluated at the spheroid's centre. The angular velocity given by (52) is not a constant: it varies as the orientation angle θ changes. This angular velocity also depends on the eccentricity e of the given spheroid, but is independent of its actual size. The angular velocity reaches its maximum value of $2Kh/(2-e^2)$ at $\theta = \frac{1}{2}\pi$ and its minimum value of $2Kh(1-e^2)/(2-e^2)$ at $\theta = 0$. In the limiting case of an elongated rod ($e \to 1$), the

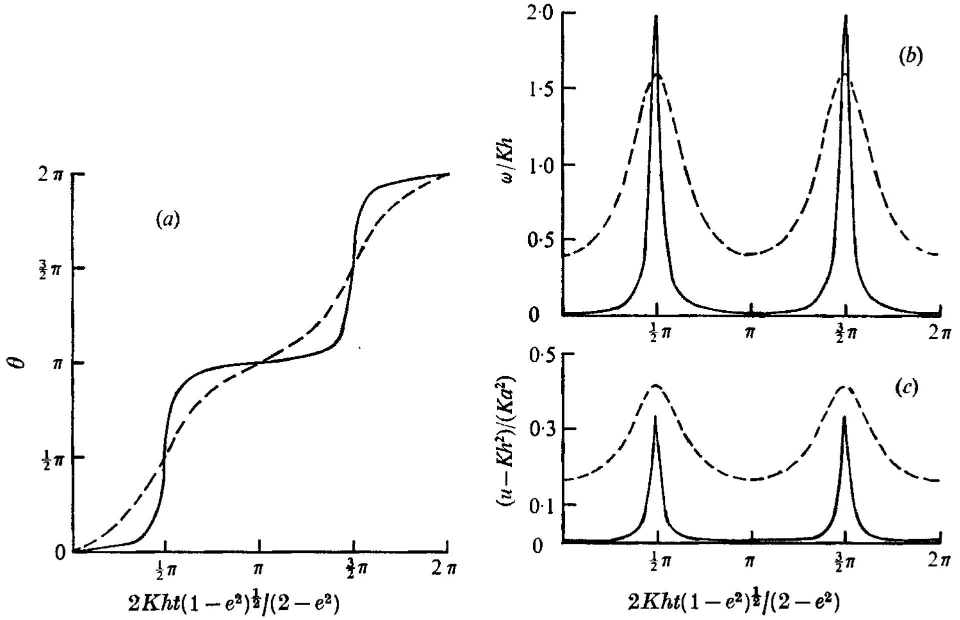

FIGURE 3. (*a*) The angular position θ, (*b*) the dimensionless angular velocity ω/Kh and (*c*) the dimensionless translational velocity $(u-Kh^2)/(Ka^2)$ *vs.* the dimensionless time $2Kht(1-e^2)^{\frac{1}{2}}/(2-e^2)$ over a complete cycle for a paraboloidal flow past a prolate spheroid. Minor-to-major axis ratio, b/a: ——, 0·1; — —, 0·5.

angular velocity attains a maximum value of $2Kh$ at $\theta = \frac{1}{2}\pi$, and vanishes like $2Khb^2/a^2$ at $\theta = 0$. It is of interest to note that the maximum angular velocity $2Kh$ of a very elongated rod is exactly the vorticity at the centre of the rod. At $\theta = 0$, the motion becomes steady for an elongated rod because of the vanishing angular velocity. In the limiting case of a sphere ($e \to 0$), at the other extreme, the angular velocity becomes constant, having a value of Kh, which is precisely half the vorticity at the centre of the sphere. There is no rotation whatsoever for a prolate spheroid of arbitrary axis ratio if its centre lies on the ξ axis ($h = 0$).

The magnitude ω of the angular velocity is related to the angle θ and time t by $\omega = d\theta/dt$. Equation (52) can be integrated to give

$$\theta = \tan^{-1}\{(1-e^2)^{\frac{1}{2}} \tan[2Kht(1-e^2)^{\frac{1}{2}}/(2-e^2)]\}, \tag{54}$$

with the initial condition $\theta = 0$ at $t = 0$. The angular position θ is plotted *vs.* the dimensionless time $2Kht(1-e^2)^{\frac{1}{2}}/(2-e^2)$ in figure 3(*a*) for a complete period between 0 and 2π. It is noted from figure 3(*a*) that as the minor-to-major axis ratio b/a becomes small, i.e. as the spheroid becomes slender, the angular position θ behaves like a step function with step discontinuities around the stationary values 0, π and 2π of θ. The dimensionless angular velocity ω/Kh given by (52) is plotted in figure 3(*b*) against the dimensionless time $2Kht(1-e^2)^{\frac{1}{2}}/(2-e^2)$. We notice from figure 3(*b*) that, as the spheroid becomes slender, the angular velocity behaves like a delta function with peaks situated at times equal to $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$. This type of motion of a very elongated rod may be described as 'flip' motion.

Equation (53) does not occur in uniform flow or linear flows; it is characteristic of paraboloidal flows. We note first from (53) that this velocity is in the ξ direction. Hence the spheroidal particle translates along a straight path parallel to the main flow, without any side drift or migration. Second, we note that this velocity is not constant, having its maximum value of $Kh^2+\frac{1}{3}Ka^2(2-e^2)$ at $\theta=\frac{1}{2}\pi$ and its minimum value of $Kh^2+\frac{2}{3}Ka^2(1-e^2)$ at $\theta=0$. Therefore the spheroidal particle *jerks* along its straight path. This 'jerk' motion is present even if the centre of the spheroid lies on the ξ axis ($h=0$). The magnitude of this jerk motion depends on the actual size a as well as the eccentricity e of the spheroid. In the limiting case of a sphere ($e\to 0$), the translational velocity becomes constant, having a value of $Kh^2+\frac{2}{3}Ka^2$, which is the same as the surface-average velocity over the spherical surface. On the other hand, as the spheroid becomes very elongated ($e\to 1$), the translational velocity reaches a maximum value of $Kh^2+\frac{1}{3}Ka^2$ at $\theta=\frac{1}{2}\pi$ (flow normal to the body axis) and a minimum value of $Kh^2+\frac{2}{3}Kb^2$ at $\theta=0$ (flow tangential to the body axis). Therefore when the flow is tangential to the axis of a very elongated rod, the rod is equivalent to a sphere of radius b, the maximum radius of the rod. When the flow is normal to the axis of a very elongated rod, it is equivalent to a sphere of radius $a/\sqrt{2}$, about 0·707 times the half-length of the rod. When the centre of an elongated rod lies on the ξ axis and the radius of the rod is very small, the velocity becomes steady at $\theta=0$, vanishing like $\frac{2}{3}Kb^2$.

In figure 3 (*c*), the translational velocity $(u-Kh^2)/(Ka^2)$ is plotted *vs*. the dimensionless time $2Kht(1-e^2)^{\frac{1}{2}}/(2-e^2)$ over a complete cycle for $b/a=0{\cdot}1$ and 0·5. We notice from this figure that the translational velocity behaves like a delta function when the spheroid becomes very slender ($b/a \ll 1$). This is precisely the characteristic of the 'jerk' motion.

We may extend the above result to a more general case. Suppose that the paraboloidal flow is still given by (43*a*) in a co-ordinate system (ξ,η,ζ) fixed in space where the η axis is chosen such that it is parallel to the shortest line segment connecting the centre of a spheroidal particle and a point on the ξ axis. Let us denote the shortest distance between the spheroid's centre and the ξ axis by h. The surface of the spheroid is again defined by (2*a*) in the body co-ordinate system (x, y, z), which is related to the fixed co-ordinate system (ξ,η,ζ) by the three Euler angles θ, ϕ and ψ such that

$$\begin{pmatrix}\xi\\ \eta+h\\ \zeta\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\cos\psi & \sin\theta\sin\psi\\ \sin\theta\cos\phi & \cos\theta\cos\phi\cos\psi-\sin\phi\sin\psi & -\cos\theta\cos\phi\sin\psi-\sin\phi\cos\psi\\ \sin\theta\sin\phi & \cos\theta\sin\phi\cos\psi+\cos\phi\sin\psi & -\cos\theta\sin\phi\sin\psi+\cos\phi\cos\psi\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}. \quad (55)$$

In other words, if we place the origin of the (ξ,η,ζ) co-ordinates at the spheroid's centre, θ is the angle between the x and ξ axes, ϕ the angle between the ξ, η and ξ, x planes and ψ the angle between the ξ, x and x, y planes. Omitting the details, we find that the translational velocity of the spheroid is still governed by (53).

However the resultant rotational motion of the spheroid is given by the following equations:

$$\dot{\phi}\cos\theta + \dot{\psi} = Kh\sin\theta\sin\phi, \tag{56a}$$

$$\dot{\theta} = \frac{2Kh}{2-e^2}(1-e^2\cos^2\theta)\cos\phi, \tag{56b}$$

$$\dot{\phi} = -\frac{2Kh}{2-e^2}(1-e^2)\cot\theta\sin\phi, \tag{56c}$$

which are related to the angular velocities about the body axes x, y and z by

$$\omega_x = \dot{\phi}\cos\theta + \dot{\psi}, \tag{57a}$$

$$\omega_y = \dot{\theta}\sin\psi - \dot{\phi}\sin\theta\cos\psi, \tag{57b}$$

$$\omega_z = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi. \tag{57c}$$

We note that (56) and (57) are the Jeffery orbital equations corresponding to a shear flow $-2Kh\eta\mathbf{e}_\xi$ with the rate of shear given by the vorticity at the centre of the spheroid. Equations (56) and (57) reduce to (52) when $\phi = 0$ and $\psi = 0$.

7. Conclusions

We have given in this paper exact solutions for a prolate spheroid in various quadratic flows at low Reynolds numbers. All exact solutions were constructed by the singularity method. Regarding the singularity method in general, we may observe the following.

(i) In addition to the five rules proposed by Chwang & Wu (1975), we find that, if the solution for a flow of reduced degree $\partial\mathbf{U}/\partial x_i$ $(i = 1, 2, 3)$ requires certain singularities, then the x_i derivatives of these singularities are needed to construct the solution associated with the flow $\mathbf{U}$.

(ii) The flow alone determines the necessary types of singularities.

(iii) The distribution range of all singularities seems to depend on the body geometry only. The distribution range determined for a plane-symmetric body in a potential flow (Wu & Chwang 1974) may also be valid in all types of Stokes flows.

(iv) For a prolate spheroid, exact solutions demand a constant density for Stokeslets, a parabolic density for rotlets, stresslets and potential doublets, a quartic density for various Stokes quadrupoles and potential quadrupoles and a sextic density for Stokes octupoles or potential octupoles, etc.

As for the motion of a spheroidal particle in a paraboloidal flow, we note the following.

(v) A spheroidal particle rotates about all three principal axes with angular velocities which are determined completely by a set of Jeffery orbital equations [(56) and (57)].

(vi) If the fluid is unbounded and its inertia neglected, the centre of the spheroidal particle moves along a straight path, parallel to the main flow direction, without any side drift or migration.

(vii) Along this straight path, the spheroid moves with a variable speed depending on its instantaneous orientation with respect to the fluid. This 'jerk' motion is governed by a trajectory equation (53) which is size dependent.

(viii) Side drift may take place when the particle is deformable or the fluid is bounded by walls or when the inertial effects are taken into account.

Although the exact solutions obtained in this paper are for prolate spheroids, they may also be valid for oblate spheroids because of their analyticity.

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REFERENCES

BATCHELOR, G. K. 1970 The stress system in a suspension of force-free particles. *J. Fluid Mech.* **41**, 545–570.

CHWANG, A. T. & WU, T. Y. 1974 Hydromechanics of low-Reynolds-number flow. Part 1. Rotation of axisymmetric prolate bodies. *J. Fluid Mech.* **63**, 607–622.

CHWANG, A. T. & WU, T. Y. 1975 Hydromechanics of low-Reynolds-number flow. Part 2. Singularity method for Stokes flows. *J. Fluid Mech.* **67**, 787–815.

EDWARDES, D. 1892 Steady motion of a viscous liquid in which an ellipsoid is constrained to rotate about a principal axis. *Quart. J. Math.* **26**, 70–78.

HANCOCK, G. J. 1953 The self-propulsion of microscopic organisms through liquids. *Proc. Roy. Soc.* A **217**, 96–121.

JEFFERY, G. B. 1922 The motion of ellipsoidal particles immersed in a viscous fluid. *Proc. Roy. Soc.* A **102**, 161–179.

OBERBECK, A. 1876 Ueber stationäre Flüssigkeitsbewegungen mit Berücksichtigung der inneren Reibung. *J. reine angew. Math.* **81**, 62–80.

SIMHA, R. 1936 Untersuchungen über die Viskosität von Suspensionen und Lösungen. *Kolloid Z.* **76**, 16–19.

WU, T. Y. & CHWANG, A. T. 1974 Double-model flow theory – a new look at the classica problem. *Proc. 10th ONR Symp. Naval Hydrodyn., Cambridge, Mass.*